The analytic continuation of the high–energy quark–quark scattering amplitude

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Abstract

It is known that the high–energy quark–quark scattering amplitude can be described by the expectation value of two lightlike Wilson lines, running along the classical trajectories of the two colliding particles. Generalizing the results of a previous paper, we give here the general proof that the expectation value of two infinite Wilson lines, forming a certain hyperbolic angle in Minkowski space–time, and the expectation value of two infinite Euclidean Wilson lines, forming a certain angle in Euclidean four–space, are connected by an analytic continuation in the angular variables. This result could be used to evaluate the high–energy scattering amplitude directly on the lattice.

1. Introduction

It is well known that the quark–quark scattering amplitude, at high squared energies s in the center of mass and small squared transferred momentum t (that is $s \to \infty$ and $|t| \ll s$, let us say $|t| \le 1$ GeV^2), can be described by the expectation value of two lightlike Wilson lines, running along the classical trajectories of the two colliding particles [1] [2].

In the center–of–mass reference system (c.m.s.), taking for example the initial trajectories of the two quarks along the x^1–axis, the scattering amplitude has the following form [explicitly indicating the color indices $(i, j, \ldots)$ and the spin indices $(\alpha, \beta, \ldots)$ of the quarks]

$$\begin{aligned} M_{fi} &= \langle \psi_{i\alpha}(p'_1)\psi_{k\gamma}(p'_2)|M|\psi_{j\beta}(p_1)\psi_{l\delta}(p_2)\rangle \\ &\underset{s\to\infty}{\sim} -\frac{i}{Z_\psi^2} \cdot \delta_{\alpha\beta}\delta_{\gamma\delta} \cdot 2s \int d^2\mathbf{z}_t e^{i\mathbf{q}\cdot\mathbf{z}_t} \langle [W_1(z_t) - \mathbf{1}]_{ij} [W_2(0) - \mathbf{1}]_{kl}\rangle \ , \end{aligned} \tag{1.1}$$

where $q = (0, 0, \mathbf{q})$, with $t = q^2 = -\mathbf{q}^2$, is the tranferred four–momentum and $z_t = (0, 0, \mathbf{z}_t)$, with $\mathbf{z}_t = (z^2, z^3)$, is the distance between the two trajectories in the *transverse* plane [the coordinates (x^0, x^1) are often called *longitudinal* coordinates]. The expectation value $\langle f(A)\rangle$ is the average of $f(A)$ in the sense of the functional integration over the gluon field A^μ (including also the determinant of the fermion matrix, i.e., $\det[i\gamma^\mu D_\mu - m]$, where $D^\mu = \partial^\mu + igA^\mu$ is the covariant derivative) [1] [2]. The two lightlike Wilson lines $W_1(z_t)$ and $W_2(0)$ in Eq. (1.1) are defined as

$$\begin{aligned} W_1(z_t) &= P \exp\left[-ig \int_{-\infty}^{+\infty} A_\mu(z_t + p_1\tau) p_1^\mu d\tau \right] \ ; \\ W_2(0) &= P \exp\left[-ig \int_{-\infty}^{+\infty} A_\mu(p_2\tau) p_2^\mu d\tau \right] \ , \end{aligned} \tag{1.2}$$

where P stands for "*path ordering*" and $A_\mu = A_\mu^a T^a$; $p_1 \simeq (E, E, \mathbf{0}_t)$ and $p_2 \simeq (E, -E, \mathbf{0}_t)$ are the initial four–momenta of the two quarks. The space–time configuration of these two Wilson lines is shown in Fig. 1.

Finally, Z_ψ in Eq. (1.1) is the fermion–field renormalization constant, which can be written in the eikonal approximation as [1]

$$Z_\psi \simeq \frac{1}{N_c}\langle \mathrm{Tr}[W_1(z_t)]\rangle = \frac{1}{N_c}\langle \mathrm{Tr}[W_1(0)]\rangle = \frac{1}{N_c}\langle \mathrm{Tr}[W_2(0)]\rangle \ , \tag{1.3}$$

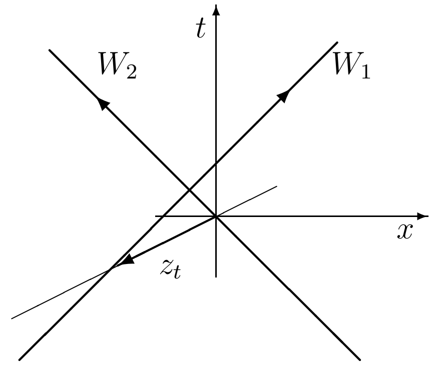

Fig. 1. The space–time configuration of the two lightlike Wilson lines W_1 and W_2 entering in the expression (1.1) for the high–energy quark–quark elastic scattering amplitude.

where N_c is the number of colours.

In what follows, we shall deal with the quantity

$$g_{M(ij,kl)}(s;\ t) \equiv \frac{1}{Z_\psi^2} \int d^2\mathbf{z}_t e^{i\mathbf{q}\cdot\mathbf{z}_t} \langle [W_1(z_t) - \mathbf{1}]_{ij} [W_2(0) - \mathbf{1}]_{kl} \rangle \ , \tag{1.4}$$

in terms of which the scattering amplitude can be written as

$$M_{fi} = \langle \psi_{i\alpha}(p_1')\psi_{k\gamma}(p_2')|M|\psi_{j\beta}(p_1)\psi_{l\delta}(p_2)\rangle \underset{s\to\infty}{\sim} -i \cdot 2s \cdot \delta_{\alpha\beta}\delta_{\gamma\delta} \cdot g_{M(ij,kl)}(s;\ t) \ . \tag{1.5}$$

The quantity $g_{M(ij,kl)}(s;\ t)$ depends not only on $t = -\mathbf{q}^2$, but also on s. In fact, as was pointed out by Verlinde and Verlinde in [3], it is a singular limit to take the Wilson lines in (1.4) exactly lightlike. A way to regularize this sort of "infrared" divergence (so called because it essentially comes from the limit $m \to 0$, where m is the quark mass) consists in letting each line have a small timelike component, so that they coincide with the classical trajectories for quarks with a finite mass m (see also Ref. [4] for a discussion about this point). In other words, one first evaluates the quantity $g_{M(ij,kl)}(\beta;\ t)$ for two Wilson lines along the trajectories of two quarks (with mass m) moving with velocity β and $-\beta$ ($0 < \beta < 1$) in the x^1–direction. This is equivalent to consider two infinite Wilson lines forming a certain (finite) hyperbolic angle χ in Minkowski space–time. Then, to obtain the correct high–energy scattering amplitude, one has to perform the limit $\beta \to 1$, that

is $\chi \to \infty$, in the expression for $g_{M(ij,kl)}(\beta;\ t)$:

$$M_{fi} = \langle \psi_{i\alpha}(p_1') \psi_{k\gamma}(p_2') | M | \psi_{j\beta}(p_1) \psi_{l\delta}(p_2) \rangle \underset{s\to\infty}{\sim} -i \cdot 2s \cdot \delta_{\alpha\beta} \delta_{\gamma\delta} \cdot g_{M(ij,kl)}(\beta \to 1;\ t) \ . \quad (1.6)$$

Proceeding in this way one obtains a $\ln s$ dependence of the amplitude, as expected from ordinary perturbation theory and as confirmed by the experiments on hadron–hadron scattering processes [5] [6]. In Sect. 3 of Ref. [4] we have followed this procedure to explicitly evaluate the second member of (1.6) up to the fourth order in the expansion in the renormalized coupling constant: the results so derived are in agreement with those obtained from ordinary perturbation theory.

The direct evaluation of the expectation value (1.4) is a highly non–trivial matter and it is strictly connected with the ultraviolet properties of Wilson–line operators [7] [8]. Some non–perturbative approaches for the calculation of (1.4) have been proposed in Refs. [9] and [10].

In a recent paper [4] we have proposed a new approach, which consists in adapting the scattering amplitude to the Euclidean world: this approach could open the way for the direct evaluation of the scattering amplitude on the lattice. More explicitly, in Ref. [4] we have given arguments showing that the expectation value of two infinite Wilson lines, forming a certain hyperbolic angle in Minkowski space–time, and the expectation value of two infinite Euclidean Wilson lines, forming a certain angle in Euclidean four–space, are likely to be connected by an analytic continuation in the angular variables. This relation of analytic continuation has been proven in Ref. [4] for an Abelian gauge theory (QED) in the so–called *quenched* approximation and for a non–Abelian gauge theory (QCD) up to the fourth order in the renormalized coupling constant in perturbation theory: a general proof was missing up to now.

In this paper, we shall generalize the results of Ref. [4] and give the rigorous proof of the above–mentioned relation of analytic continuation for a non–Abelian gauge theory with gauge group $SU(N_c)$ [as well as for an Abelian gauge theory (QED)]. The approach adopted in Ref. [4] consisted in explicitly evaluating the amplitudes $g_M(\chi;\ t)$ and $g_E(\theta;\ t)$, in the Minkowski and the Euclidean world, in some given approximation (such as the *quenched* approximation) or up to some order in perturbation theory and in finally comparing the two expressions so obtained. Instead, in this paper we shall give a general proof, which essentially exploits the relation between the gluonic Green functions in the two theories.

2. From Minkowskian to Euclidean theory

Let us consider the following quantity, defined in Minkowski space–time:

$$
\begin{aligned}
g_M(p_1, p_2;\ t) &= \frac{M(p_1, p_2;\ t)}{Z_W^2} , \\
M(p_1, p_2;\ t) &= \int d^2\mathbf{z}_t e^{i\mathbf{q}\cdot\mathbf{z}_t} \langle [W_1(z_t) - \mathbf{1}]_{ij} [W_2(0) - \mathbf{1}]_{kl} \rangle ,
\end{aligned}
\tag{2.1}
$$

where p_1 and p_2 are the four–momenta [lying (for example) in the plane (x^0, x^1)], which define the trajectories of the two Wilson lines W_1 and W_2 ($A_\mu = A_\mu^a T^a$ and m is the fermion mass):

$$
\begin{aligned}
W_1(z_t) &\equiv P \exp \left[-ig \int_{-\infty}^{+\infty} A_\mu (z_t + \frac{p_1}{m}\tau) \frac{p_1^\mu}{m} d\tau \right] ; \\
W_2(0) &\equiv P \exp \left[-ig \int_{-\infty}^{+\infty} A_\mu (\frac{p_2}{m}\tau) \frac{p_2^\mu}{m} d\tau \right] .
\end{aligned}
\tag{2.2}
$$

Z_W in Eq. (2.1) is defined as (N_c being the number of colours)

$$
Z_W \equiv \frac{1}{N_c} \langle \mathrm{Tr}[W_1(z_t)] \rangle = \frac{1}{N_c} \langle \mathrm{Tr}[W_1(0)] \rangle = \frac{1}{N_c} \langle \mathrm{Tr}[W_2(0)] \rangle .
\tag{2.3}
$$

(The two last equalities come from the Poincaré invariance.) This is a sort of Wilson–line's renormalization constant: as shown in Ref. [1], Z_W coincides with the fermion renormalization constant Z_ψ in the eikonal approximation.

By virtue of the Lorentz symmetry, we can define p_1 and p_2 in the c.m.s. of the two particles, moving with speed β and $-\beta$ along the x^1–direction:

$$
\begin{aligned}
p_1^\mu &= E(1, \beta, \mathbf{0}_t) , \\
p_2^\mu &= E(1, -\beta, \mathbf{0}_t) ,
\end{aligned}
\tag{2.4}
$$

where $E = m/\sqrt{1-\beta^2}$ (in units with $c = 1$) is the energy of each particle (so that: $s = 4E^2$).

We now introduce the hyperbolic angle ψ [in the plane (x^0, x^1)] of the trajectory of W_1: it is given by $\beta = \tanh\psi$. We can give the explicit form of the Minkowski four–vectors $u_1 = p_1/m$ and $u_2 = p_2/m$ in terms of the hyperbolic angle ψ:

$$
\begin{aligned}
u_1 = \frac{p_1}{m} &= (\cosh\psi, \sinh\psi, \mathbf{0}_t) , \\
u_2 = \frac{p_2}{m} &= (\cosh\psi, -\sinh\psi, \mathbf{0}_t) .
\end{aligned}
\tag{2.5}
$$

Clearly, $u_1^2 = u_2^2 = 1$ and

$$u_1 \cdot u_2 = \cosh(2\psi) = \cosh\chi \ , \tag{2.6}$$

where $\chi = 2\psi$ is the hyperbolic angle [in the plane (x^0, x^1)] between the two trajectories of W_1 and W_2.

In an analogous way, we can consider the following quantity, defined in Euclidean space–time:

$$\begin{aligned} g_E(p_{1E}, p_{2E};\ t) &= \frac{E(p_{1E}, p_{2E};\ t)}{Z_{WE}^2} \ , \\ E(p_{1E}, p_{2E};\ t) &= \int d^2\mathbf{z}_t e^{i\mathbf{q}\cdot\mathbf{z}_t} \langle [W_{1E}(z_{tE}) - \mathbf{1}]_{ij} [W_{2E}(0) - \mathbf{1}]_{kl} \rangle_E \ , \end{aligned} \tag{2.7}$$

where $z_{tE} = (z_1, z_2, z_3, z_4) = (0, \mathbf{z}_t, 0)$ and $q_E = (0, \mathbf{q}, 0)$ (so that: $q_E^2 = \mathbf{q}^2 = -t$). The expectation value $\langle \ldots \rangle_E$ must be intended now as a functional integration with respect to the gauge variable $A_\mu^{(E)} = A_\mu^{(E)a} T^a$ in the Euclidean theory. The Euclidean four–vectors p_{1E} and p_{2E} [lying (for example) in the plane (x_1, x_4)] define the trajectories of the two Euclidean Wilson lines W_{1E} and W_{2E}:

$$\begin{aligned} W_{1E}(z_{tE}) &\equiv P \exp\left[-ig \int_{-\infty}^{+\infty} A_\mu^{(E)}(z_{tE} + p_{1E}\tau) p_{1E\mu} d\tau \right] \ ; \\ W_{2E}(0) &\equiv P \exp\left[-ig \int_{-\infty}^{+\infty} A_\mu^{(E)}(p_{2E}\tau) p_{2E\mu} d\tau \right] \ . \end{aligned} \tag{2.8}$$

Z_{WE} in Eq. (2.7) is defined analogously to Z_W in Eq. (2.3):

$$Z_{WE} \equiv \frac{1}{N_c} \langle \mathrm{Tr}[W_{1E}(z_{tE})] \rangle = \frac{1}{N_c} \langle \mathrm{Tr}[W_{1E}(0)] \rangle = \frac{1}{N_c} \langle \mathrm{Tr}[W_{2E}(0)] \rangle \ . \tag{2.9}$$

(The two last equalities come from the $O(4)$ *plus* translation invariance.)

We can now expand the Wilson lines W_1 and W_2 in power series of the coupling constant g and take the pieces with g^n and g^r respectively. Their contribution to the amplitude $M(p_1, p_2;\ t)$ will be called $M_{(n,r)}(p_1, p_2;\ t)$ (so that $M = \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} M_{(n,r)}$) and is given by

$$\begin{aligned} & M_{(n,r)}(p_1, p_2;\ t) = (-ig)^{(n+r)} (T^{a_1} \ldots T^{a_n})_{ij} (T^{b_1} \ldots T^{b_r})_{kl} \int d^2\mathbf{z}_t e^{i\mathbf{q}\cdot\mathbf{z}_t} \times \\ & \int d\tau_1 \frac{p_1^{\mu_1}}{m} \ldots \int d\tau_n \frac{p_1^{\mu_n}}{m} \int d\omega_1 \frac{p_2^{\nu_1}}{m} \ldots \int d\omega_r \frac{p_2^{\nu_r}}{m} \times \\ & \theta(\tau_n - \tau_{n-1}) \ldots \theta(\tau_2 - \tau_1) \theta(\omega_r - \omega_{r-1}) \ldots \theta(\omega_2 - \omega_1) \times \\ & \langle A_{\mu_1}^{a_1}(z_t + \frac{p_1}{m}\tau_1) \ldots A_{\mu_n}^{a_n}(z_t + \frac{p_1}{m}\tau_n) A_{\nu_1}^{b_1}(\frac{p_2}{m}\omega_1) \ldots A_{\nu_r}^{b_r}(\frac{p_2}{m}\omega_r) \rangle \ . \end{aligned} \tag{2.10}$$

The corresponding quantity for the Euclidean theory $E_{(n,r)}(p_{1E}, p_{2E};\ t)$, obtained taking the pieces with g^n and g^r in the expansion of the Euclidean Wilson lines W_{1E} and W_{2E} inside $E(p_{1E}, p_{2E};\ t)$, is given by

$$\begin{aligned}
E_{(n,r)}(p_{1E}, p_{2E};\ t) &= (-ig)^{(n+r)}(T^{a_1}\dots T^{a_n})_{ij}(T^{b_1}\dots T^{b_r})_{kl}\int d^2\mathbf{z}_t e^{i\mathbf{q}\cdot\mathbf{z}_t}\times \\
&\int d\tau_1 p_{1E}^{\mu_1}\dots\int d\tau_n p_{1E}^{\mu_n}\int d\omega_1 p_{2E}^{\nu_1}\dots\int d\omega_r p_{2E}^{\nu_r}\theta(\tau_n-\tau_{n-1})\dots\theta(\tau_2-\tau_1)\times \\
&\theta(\omega_r-\omega_{r-1})\dots\theta(\omega_2-\omega_1)\langle A^{a_1}_{(E)\mu_1}(z_{tE}+p_{1E}\tau_1)\dots A^{a_n}_{(E)\mu_n}(z_{tE}+p_{1E}\tau_n)\times \\
&\times A^{b_1}_{(E)\nu_1}(p_{2E}\omega_1)\dots A^{b_r}_{(E)\nu_r}(p_{2E}\omega_r)\rangle_E\ . \qquad (2.11)
\end{aligned}$$

It is known that, making use of the correspondence

$$\begin{aligned}
&A_0(x)\to iA_4^{(E)}(x_E) \quad , \quad A_k(x)\to A_k^{(E)}(x_E) \\
&\text{with}:\ x^0\to -ix_{E4} \quad , \quad \mathbf{x}\to\mathbf{x}_E\ , \qquad (2.12)
\end{aligned}$$

between the Minkowski and the Euclidean world, the following relationship is derived between the gluonic Green functions in the two theories:

$$\begin{aligned}
&\tilde{B}^{\mu_1}_{(1)}\dots\tilde{B}^{\mu_N}_{(N)}\langle A^{a_1}_{\mu_1}(\tilde{x}_{(1)})\dots A^{a_N}_{\mu_N}(\tilde{x}_{(N)})\rangle = \\
&\quad = B_{(1)E\mu_1}\dots B_{(N)E\mu_N}\langle A^{a_1}_{(E)\mu_1}(x_{(1)E})\dots A^{a_N}_{(E)\mu_N}(x_{(N)E})\rangle_E\ , \qquad (2.13)
\end{aligned}$$

where $x_{(i)E} = (\mathbf{x}_{(i)E}, x_{(i)E4})$ are Euclidean four–coordinates and $B_{(i)E} = (\mathbf{B}_{(i)E}, B_{(i)E4})$ are any Euclidean four–vectors, while $\tilde{x}_{(i)}$ and $\tilde{B}_{(i)}$ are Minkowski four–vectors defined as

$$\begin{aligned}
\tilde{x}_{(i)} = (\tilde{x}^0_{(i)}, \tilde{\mathbf{x}}_{(i)}) &= (-ix_{(i)E4}, \mathbf{x}_{(i)E})\ , \\
\tilde{B}_{(i)} = (\tilde{B}^0_{(i)}, \tilde{\mathbf{B}}_{(i)}) &= (-iB_{(i)E4}, \mathbf{B}_{(i)E})\ . \qquad (2.14)
\end{aligned}$$

For example, in the case $N = 2$, if one defines the gluonic propagators as

$$\begin{aligned}
G^{ab}_{\mu\nu}(x,y) &\equiv \langle A^a_\mu(x)A^b_\nu(y)\rangle\ , \\
G^{ab}_{(E)\mu\nu}(x_E,y_E) &\equiv \langle A^a_{(E)\mu}(x_E)A^b_{(E)\nu}(y_E)\rangle_E\ , \qquad (2.15)
\end{aligned}$$

one finds that

$$\begin{aligned}
G^{ab}_{00}(\tilde{x},\tilde{y}) &= -G^{ab}_{(E)44}(x_E,y_E)\ , \\
G^{ab}_{0j}(\tilde{x},\tilde{y}) &= iG^{ab}_{(E)4j}(x_E,y_E)\ , \\
G^{ab}_{j0}(\tilde{x},\tilde{y}) &= iG^{ab}_{(E)j4}(x_E,y_E)\ , \\
G^{ab}_{jk}(\tilde{x},\tilde{y}) &= G^{ab}_{(E)jk}(x_E,y_E)\ , \qquad (2.16)
\end{aligned}$$

where $j,k=1,2,3$ are indices for the spatial components and $\tilde{x}$ and $\tilde{y}$ are defined as in Eq. (2.14). From these relations, one immediately derives Eq. (2.13) for $N=2$, with $\tilde{B}$ defined as in Eq. (2.14). The result can be trivially generalized to every N.

In our specific case, we can use Eq. (2.13) to state that

$$\begin{aligned} &\frac{\tilde{p}_1^{\mu_1}}{m}\dots\frac{\tilde{p}_1^{\mu_n}}{m}\frac{\tilde{p}_2^{\nu_1}}{m}\dots\frac{\tilde{p}_2^{\nu_r}}{m}\langle A_{\mu_1}^{a_1}(z_t+\frac{\tilde{p}_1}{m}\tau_1)\dots A_{\mu_n}^{a_n}(z_t+\frac{\tilde{p}_1}{m}\tau_n)A_{\nu_1}^{b_1}(\frac{\tilde{p}_2}{m}\omega_1)\dots A_{\nu_r}^{b_r}(\frac{\tilde{p}_2}{m}\omega_r)\rangle = \\ &=\frac{p_{1E}^{\mu_1}}{m}\dots\frac{p_{1E}^{\mu_n}}{m}\frac{p_{2E}^{\nu_1}}{m}\dots\frac{p_{2E}^{\nu_r}}{m}\langle A_{(E)\mu_1}^{a_1}(z_{tE}+\frac{p_{1E}}{m}\tau_1)\dots A_{(E)\mu_n}^{a_n}(z_{tE}+\frac{p_{1E}}{m}\tau_n)\times \\ &\times A_{(E)\nu_1}^{b_1}(\frac{p_{2E}}{m}\omega_1)\dots A_{(E)\nu_r}^{b_r}(\frac{p_{2E}}{m}\omega_r)\rangle_E\ , \end{aligned} \tag{2.17}$$

where $p_{iE}=(\mathbf{p}_{iE},p_{iE4})$, for $i=1,2$, are two Euclidean four–vectors and $\tilde{p}_i$ are the two corresponding Minkowski four–vectors, obtained according to Eq. (2.14):

$$\tilde{p}_i=(\tilde{p}_i^0,\tilde{\mathbf{p}}_i)=(-ip_{iE4},\mathbf{p}_{iE})\ . \tag{2.18}$$

By virtue of the definitions (2.10) and (2.11) for $M_{(n,r)}$ and $E_{(n,r)}$ respectively, Eq. (2.17) implies that:

$$E_{(n,r)}(\frac{p_{1E}}{m},\frac{p_{2E}}{m};\ t)=M_{(n,r)}(\tilde{p}_1,\tilde{p}_2;\ t)\ . \tag{2.19}$$

This relation is valid for every couple of integer numbers (n,r), so that, more generally:

$$E(\frac{p_{1E}}{m},\frac{p_{2E}}{m};\ t)=M(\tilde{p}_1,\tilde{p}_2;\ t)\ . \tag{2.20}$$

Of course M, considered as a general function of p_1, p_2 [and $q=(0,0,\mathbf{q})$], can only depend on the scalar quantities constructed with the vectors p_1, p_2 and $q=(0,0,\mathbf{q})$: the only possibilities are $q^2=-\mathbf{q}^2=t$, $p_1\cdot p_2$, p_1^2 and p_2^2, since $p_1\cdot q=p_2\cdot q=0$. Moreover, it is clear from the definitions (2.1) and (2.2) that M cannot depend on the (positive) normalizations of the four–vectors p_1 and p_2: in other words, we obtain the same result for M if we substitute (p_1,p_2) with $(\alpha_1 p_1,\alpha_2 p_2)$, α_1 and α_2 being arbitrary positive constants.

Therefore, M is forced to have the following form:

$$M(p_1,p_2;\ t)=f_M\left(\frac{p_1}{\sqrt{p_1^2}}\cdot\frac{p_2}{\sqrt{p_2^2}};\ t\right)\ . \tag{2.21}$$

For analogous reasons, E must be of the form:

$$E(p_{1E}, p_{2E};\ t) = f_E\left(\frac{p_{1E}}{|p_{1E}|} \cdot \frac{p_{2E}}{|p_{2E}|};\ t\right)\ , \tag{2.22}$$

where $|B_E| \equiv \sqrt{\sum_{\mu=1}^{4} B_{E\mu}^2}$ is the Euclidean norm. (A short remark about the notation: we have denoted everywhere the scalar product by a "$\cdot$", both in the Minkowski and the Euclidean world. Of course, when A and B are Minkowski four–vectors, then $A \cdot B = A^\mu B_\mu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}$; while, if A_E and B_E are Euclidean four–vectors, then $A_E \cdot B_E = A_{E\mu} B_{E\mu} = \mathbf{A}_E \cdot \mathbf{B}_E + A_{E4} B_{E4}$.) Therefore, the relation (2.20) can be re–formulated as follows [observing that $(p_{iE}/m)/|(p_{iE}/m)| = p_{iE}/|p_{iE}|$]

$$f_E(v_{1E} \cdot v_{2E};\ t) = f_M(\bar{u}_1 \cdot \bar{u}_2;\ t)\ , \tag{2.23}$$

where v_{1E} and v_{2E} are the Euclidean four–versors corresponding to p_{1E} and p_{2E} ($v_{1E}^2 = v_{2E}^2 = 1$):

$$v_{1E} = \frac{p_{1E}}{|p_{1E}|} \quad , \quad v_{2E} = \frac{p_{2E}}{|p_{2E}|}\ , \tag{2.24}$$

while $\bar{u}_1$ and $\bar{u}_2$ are the Minkowski four–vectors defined as

$$\bar{u}_1 = \frac{\tilde{p}_1}{\sqrt{\tilde{p}_1^2}} \quad , \quad \bar{u}_2 = \frac{\tilde{p}_2}{\sqrt{\tilde{p}_2^2}}\ . \tag{2.25}$$

(It is clear that: $\bar{u}_1^2 = \bar{u}_2^2 = 1$.) By virtue of the $O(4)$ symmetry of the Euclidean theory, we can choose a reference frame in which the spatial vectors $\mathbf{v}_{1E}$ and $\mathbf{v}_{2E} = -\mathbf{v}_{1E}$ are along the x_1–direction. The two four–momenta v_{1E} and v_{2E} are, therefore,

$$\begin{aligned} v_{1E} &= (\sin\phi, \mathbf{0}_t, \cos\phi)\ ; \\ v_{2E} &= (-\sin\phi, \mathbf{0}_t, \cos\phi)\ , \end{aligned} \tag{2.26}$$

where ϕ is the angle formed by each trajectory with the x_4–axis. The value of ϕ is between 0 and $\pi/2$, so that the angle $\theta = 2\phi$ between the two Euclidean trajectories W_{1E} and W_{2E} lies in the range $[0, \pi]$: it is always possible to make such a choice by virtue of the $O(4)$ symmetry of the Euclidean theory. In such a reference frame, we can write $v_{1E} \cdot v_{2E} = \cos\theta$.

From Eq. (2.18) we have that $\tilde{p}_i^2 = -|p_{iE}|^2 < 0$ and $\sqrt{\tilde{p}_i^2} = -i|p_{iE}|$. The sign of the squared root is fixed in the following way: in the system where $\mathbf{p}_i = \mathbf{0}$, we have that $\sqrt{p_i^2} = p_i^0$ (if we take $p_i^0 > 0$). This relation is continued so to have $\sqrt{\tilde{p}_i^2} = \tilde{p}_i^0$ in the system where $\tilde{\mathbf{p}}_i = \mathbf{0}$. But $\tilde{\mathbf{p}}_i = \mathbf{p}_{iE} = \mathbf{0}$, so that $\tilde{p}_i^0 = -ip_{iE4} = -i|p_{iE}|$ (if we take $p_{iE4} > 0$). Therefore, in this particular system $\sqrt{\tilde{p}^2} = \tilde{p}^0 = -ip_{iE4} = -i|p_{iE}|$. So we take $\sqrt{\tilde{p}^2} = -i|p_{iE}|$ in every system. This implies that:

$$\bar{u}_i = \frac{\tilde{p}_i}{\sqrt{\tilde{p}_i^2}} = (v_{iE4}, i\mathbf{v}_{iE}) \ . \tag{2.27}$$

With the explicit form of v_{1E} and v_{2E} given by Eq. (2.26), we find that

$$\begin{aligned} \bar{u}_1 &= (\cos\phi, i\sin\phi, \mathbf{0}_t) \ , \\ \bar{u}_2 &= (\cos\phi, -i\sin\phi, \mathbf{0}_t) \ , \end{aligned} \tag{2.28}$$

and consequently $\bar{u}_1^2 = \bar{u}_2^2 = 1$ and

$$\bar{u}_1 \cdot \bar{u}_2 = \cos(2\phi) = \cos\theta \ . \tag{2.29}$$

A comparison with the expressions (2.5) for the Minkowski four–vectors u_1 and u_2 reveals that $\bar{u}_1$ and $\bar{u}_2$ are obtained from u_1 and u_2 after the following analytic continuation in the angular variables is made:

$$\chi \to i\theta \ . \tag{2.30}$$

(We remind that $\phi = \theta/2$ and $\psi = \chi/2$.) Therefore, by virtue of Eqs. (2.21) and (2.22), the relation (2.23) can be formulated as follows:

$$E(\theta;\ t) = M(\chi \to i\theta;\ t) \ . \tag{2.31}$$

Let us consider, now, the Wilson–line's renormalization constant Z_W:

$$Z_W \equiv \frac{1}{N_c} \langle \mathrm{Tr}[W_1(0)] \rangle \ . \tag{2.32}$$

We can expand W_1 in power series of g and take the piece with g^n, whose contribution to Z_W we call $Z_W^{(n)}$:

$$\begin{aligned} Z_W^{(n)} = \frac{(-ig)^n}{N_c} \mathrm{Tr}(T^{a_1} \dots T^{a_n}) \int d\tau_1 \frac{p_1^{\mu_1}}{m} \dots \int d\tau_n \frac{p_1^{\mu_n}}{m} \times \\ \times \theta(\tau_n - \tau_{n-1}) \dots \theta(\tau_2 - \tau_1) \langle A_{\mu_1}^{a_1}(\frac{p_1}{m}\tau_1) \dots A_{\mu_n}^{a_n}(\frac{p_1}{m}\tau_n) \rangle \ . \end{aligned} \tag{2.33}$$

In the Euclidean theory we have, analogously:

$$Z_{WE} \equiv \frac{1}{N_c} \langle \mathrm{Tr}[W_{1E}(0)] \rangle_E \ , \tag{2.34}$$

and

$$Z_{WE}^{(n)} = \frac{(-ig)^n}{N_c} \mathrm{Tr}(T^{a_1} \dots T^{a_n}) \int d\tau_1 p_{1E}^{\mu_1} \dots \int d\tau_n p_{1E}^{\mu_n} \times$$
$$\times \theta(\tau_n - \tau_{n-1}) \dots \theta(\tau_2 - \tau_1) \langle A^{a_1}_{(E)\mu_1}(p_{1E}\tau_1) \dots A^{a_n}_{(E)\mu_n}(p_{1E}\tau_n) \rangle_E \ . \tag{2.35}$$

Using Eq. (2.13), we can derive the following relation:

$$\frac{\tilde{p}_1^{\mu_1}}{m} \dots \frac{\tilde{p}_1^{\mu_n}}{m} \langle A^{a_1}_{\mu_1}(\frac{\tilde{p}_1}{m}\tau_1) \dots A^{a_n}_{\mu_n}(\frac{\tilde{p}_1}{m}\tau_n) \rangle =$$
$$= \frac{p_{1E}^{\mu_1}}{m} \dots \frac{p_{1E}^{\mu_n}}{m} \langle A^{a_1}_{(E)\mu_1}(\frac{p_{1E}}{m}\tau_1) \dots A^{a_n}_{(E)\mu_n}(\frac{p_{1E}}{m}\tau_n) \rangle_E \ , \tag{2.36}$$

where, as usual, $p_{1E} = (\mathbf{p}_{1E}, p_{1E4})$ and $\tilde{p}_1 = (\tilde{p}_1^0, \tilde{\mathbf{p}}_1) = (-ip_{1E4}, \mathbf{p}_{1E})$. If we define

$$Z_W \equiv h_M(p_1) \quad , \quad Z_W^{(n)} = h_M^{(n)}(p_1) \ ,$$
$$Z_{WE} \equiv h_E(p_{1E}) \quad , \quad Z_{WE}^{(n)} = h_E^{(n)}(p_{1E}) \ , \tag{2.37}$$

from Eq. (2.36) we obtain

$$h_E^{(n)}(\frac{p_{1E}}{m}) = h_M^{(n)}(\tilde{p}_1) \ . \tag{2.38}$$

This relation is valid for every integer number n and so we also have, more generally:

$$h_E(\frac{p_{1E}}{m}) = h_M(\tilde{p}_1) \ . \tag{2.39}$$

From the definitions (2.32) and (2.2), $h_M(p_1)$, considered as a function of a general four–vector p_1, is a scalar function constructed with the only four–vector p_1. In addition, $h_M(p_1)$ does not depend on the (positive) normalization of p_1: in other words, $h_M(\alpha p_1) = h_M(p_1)$ for every positive α. Therefore, $h_M(p_1)$ is forced to have the form

$$h_M(p_1) = H_M(u_1^2) = H_M(1) \ , \tag{2.40}$$

where $u_1 = p_1/\sqrt{p_1^2}$ ($u_1^2 = 1$). In a perfectly analogous way, for the Euclidean case we have that:

$$h_E(p_{1E}) = H_E(v_{1E}^2) = H_E(1) \ , \tag{2.41}$$

where $v_{1E} = p_{1E}/|p_{1E}|$ ($v_{1E}^2 = 1$). Therefore, the first member of Eq. (2.39) is just equal to $h_E(p_{1E}/m) = H_E(v_{1E}^2) = H_E(1)$ [observing that $(p_{1E}/m)/|(p_{1E}/m)| = p_{1E}/|p_{1E}|$], and the second member is given by $h_M(\tilde{p}_1) = H_M(\bar{u}_1^2) = H_M(1)$, where $\bar{u}_1 = \tilde{p}_1/\sqrt{\tilde{p}_1^2}$ ($\bar{u}_1^2 = 1$). Then Eq. (2.39) implies that

$$H_E(1) = H_M(1) \ . \tag{2.42}$$

That is, from Eqs. (2.37), (2.40) and (2.41):

$$Z_{WE} = Z_W \ . \tag{2.43}$$

Combining this identity with Eq. (2.31), we find the following relation between the amplitudes $g_M(\chi;\ t) = M(\chi;\ t)/Z_W^2$ and $g_E(\theta;\ t) = E(\theta;\ t)/Z_{WE}^2$:

$$\begin{aligned} & g_M(\chi;\ t) \underset{\chi\to i\theta}{\longrightarrow} g_M(i\theta;\ t) = g_E(\theta;\ t) \ ; \\ \text{or :} \ \ & g_E(\theta;\ t) \underset{\theta\to -i\chi}{\longrightarrow} g_E(-i\chi;\ t) = g_M(\chi;\ t) \ . \end{aligned} \tag{2.44}$$

We have derived the relation (2.44) of analytic continuation for a non–Abelian gauge theory with gauge group $SU(N_c)$. It is clear, from the derivation given above, that the same result is valid also for an Abelian gauge theory (QED). We have thus completely generalized the results of Ref. [4], where the same relation (2.44) had been proven for an Abelian gauge theory (QED) in the so–called *quenched* approximation and for a non–Abelian gauge theory (QCD) up to the fourth order in the renormalized coupling constant in perturbation theory. The approach adopted in Ref. [4] consisted in explicitly evaluating the amplitudes $g_M(\chi;\ t)$ and $g_E(\theta;\ t)$, in the Minkowski and the Euclidean world, in some given approximation (such as the *quenched* approximation) or up to some order in perturbation theory and in finally comparing the two expressions so obtained. Instead, in this paper we have given a general proof of Eq. (2.44), which essentially exploits the relation (2.13) between the gluonic Green functions in the two theories.

Therefore, it is possible to reconstruct the high–energy scattering amplitude by evaluating a correlation of two infinite Wilson lines forming a certain angle θ in Euclidean four–space, then by continuing this quantity in the angular variable, $\theta \to -i\chi$, where χ is the hyperbolic angle between the two Wilson lines in Minkowski space–time, and finally by performing the limit $\chi \to \infty$ (i.e., $\beta \to 1$). In fact, the high–energy scattering amplitude is given by

$$\begin{aligned} M_{fi} &= \langle \psi_{i\alpha}(p_1')\psi_{k\gamma}(p_2')|M|\psi_{j\beta}(p_1)\psi_{l\delta}(p_2)\rangle \\ &\underset{s\to\infty}{\sim} -i \cdot 2s \cdot \delta_{\alpha\beta}\delta_{\gamma\delta} \cdot g_M(\chi\to\infty;\ t) \ . \end{aligned} \tag{2.45}$$

The quantity $g_M(\chi;\ t)$, defined by Eq. (2.1) in the Minkowski world, is linked to the corresponding quantity $g_E(\theta;\ t)$, defined by Eq. (2.7) in the Euclidean world, by the analytic continuation (2.44) in the angular variables. The important thing to note here is that the quantity $g_E(\theta;\ t)$, defined in the Euclidean world, may be computed non perturbatively by well–known and well–established techniques, for example by means of the formulation of the theory on the lattice. In all cases, once one has obtained the quantity $g_E(\theta;\ t)$, one still has to perform an analytic continuation in the angular variable $\theta \to -i\chi$, and finally one has to extrapolate to the limit $\chi \to \infty$ (i.e., $\beta \to 1$). For deriving the dependence on s one exploits the fact that both β and ψ (or equivalently χ) are dependent on s. In fact, from $E = m/\sqrt{1-\beta^2}$ and from $s = 4E^2$, one immediately finds that

$$\beta = \sqrt{1 - \frac{4m^2}{s}} \ . \tag{2.46}$$

By inverting this equation and using the relation $\beta = \tanh\psi$, we derive:

$$s = 4m^2 \cosh^2\psi = 2m^2(\cosh\chi + 1) \ . \tag{2.47}$$

Therefore, in the high–energy limit $s \to \infty$ (or $\beta \to 1$), the hyperbolic angle $\chi = 2\psi$ is essentially equal to the logarithm of s (for a finite non–zero quark mass m):

$$\chi = 2\psi \underset{s\to\infty}{\sim} \ln s \ . \tag{2.48}$$

As an example, we have shown in Ref. [4] how, using this approach, one can re–derive the well–known *Regge Pole Model* [11]. Of course, the most interesting results are expected from an *exact* non perturbative approach, for example by directly computing $g_E(\theta;\ t)$ on the lattice: a considerable progress could be achieved along this direction in the near future.

References

[1] O. Nachtmann, Ann. Phys. **209**, 436 (1991).

[2] E. Meggiolaro, Phys. Rev. D **53**, 3835 (1996).

[3] H. Verlinde and E. Verlinde, Princeton University, report No. PUPT–1319 (revised 1993); hep–th/9302104.

[4] E. Meggiolaro, Pisa preprint, IFUP–TH 10/96 (1996); hep–th/9602104; Zeitschrift für Physik **C** (1997), currently in press.

[5] H. Cheng and T.T. Wu, *Expanding Protons: Scattering at High Energies* (MIT Press, Cambridge, Massachussets, 1987).

[6] L.N. Lipatov, in *Review in Perturbative QCD*, edited by A.H. Mueller (World Scientific, Singapore, 1989), and references therein.

[7] I.Ya. Aref'eva, Phys. Lett. **93B**, 347 (1980).

[8] G.P. Korchemsky, Phys. Lett. B **325**, 459 (1994); I.A. Korchemskaya and G.P. Korchemsky, Nucl. Phys. **B437**, 127 (1995).

[9] I.Ya. Aref'eva, Phys. Lett. B **325**, 171 (1994); **328**, 411 (1994).

[10] H.G. Dosch, E. Ferreira and A. Krämer, Phys. Rev. D **50**, 1992 (1994).

[11] T. Regge, Nuovo Cimento **14**, 951 (1959); **18**, 947 (1960).